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Inversion of a Covariance Matrix

Let $x(t)$ be the stationary random function, the $\bar{x}(t)$ denoting its mean value. Let us suppose that the correlation function $k(\tau)$ of the random function $x(t)$

$$k(\tau) = E\{[x(t) - \bar{x}(t)][x(t+\tau) - \bar{x}(t+\tau)]\} \tag{1}$$

is of the following special type

$$k(\tau) = k_0^2 \exp(-\tau/T), \tag{2}$$

where the k_0^2 and the T are given positive constants. The symbol E stands for the mathematical expectation.

Correlation function of the simple type (2) can take place if the function $x(t)$ is the output of the linear system of the first order the input of which is the white noise [1]. The correlation function (2) can be considered as the approximation of a more complicated correlation function, too.

One can build the $(n+1)$-dimensional random vector $\mathbf{x}$ from the observations of the random function $x(t)$ in the points t_i,

$$\mathbf{x} = \| x(t_0), x(t_1),..., x(t_n) \| . \tag{3}$$

In the case of regular sampling

$$t_i = ih, \tag{4}$$

where the h is a positive constant, the covariance matrix $\mathbf{K}$

$$\mathbf{K} = E\{\mathbf{x}'\mathbf{x}\} \tag{5}$$

will have the form

$$\mathbf{K} = k_0^2 \begin{Vmatrix} 1, & q, & q^2, & q^3,..., q^n \\ q, & 1, & q, & q^2,..., q^{n-1} \\ q^2, & q, & 1, & q,..., q^{n-2} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \;\; \cdots \;\; \cdot\cdot \\ q^n, & q^{n-1}, & q^{n-2}, & q^{n-3},..., \;\; 1 \end{Vmatrix}, \tag{6}$$

where the q stands for the expression

$$q = \exp(-h/T). \tag{7}$$

The column vector $\mathbf{x}'$ is the transposed vector $\mathbf{x}$. Since both constants h and T are positive, the matrix $\mathbf{K}$ is regular. Thus, such regular matrix $\mathbf{S}$ of the "triangular" type can be found that the equation

$$\mathbf{K} = k_0^2\mathbf{S}'\mathbf{S} \tag{8}$$

holds. It can be easily proved by the substitution into the Eq. (8) that the matrix $\mathbf{S}$ is of the following form:

$$S = \left\| \begin{matrix} q^0, & q^1, & q^2, & q^3,\dots, q^n \\ 0, & rq^0, & rq^1, & rq^2,\dots, rq^{n-1} \\ 0, & 0, & rq^0, & rq^1,\dots, rq^{n-2} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \ \cdots \ \cdot\cdot \\ 0, & 0, & 0, & 0,\dots, rq^0 \end{matrix} \right\|, \tag{9}$$

where

$$r = \sqrt{1 - q^2}. \tag{10}$$

The inverse of this matrix is

$$\mathbf{S}^{-1} = \left\| \begin{matrix} 1, & -q/r, & 0,\dots, 0, & 0 \\ 0, & 1/r, & -q/r,\dots, 0, & 0 \\ 0, & 0, & 1/r,\dots, 0, & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot \ \cdots \ \cdot\cdot & \cdot\cdot \\ 0, & 0, & 0,\dots, 0, & 0 \\ 0, & 0, & 0,\dots, 1/r, & -q/r \\ 0, & 0, & 0,\dots, 0, & 1/r \end{matrix} \right\|. \tag{11}$$

Using (8) and (11) one can get the inverse $\mathbf{K}$ of the covariance matrix

$$\mathbf{K}^{-1} = (1/k_0^2)\,\mathbf{S}^{-1}(\mathbf{S}^{-1})'$$

or

$$\mathbf{K}^{-1} = (1/k_0^2r^2) \left\| \begin{matrix} 1, & -q, & 0,\dots, 0, & 0, & 0 \\ -q, & 1+q^2, & -q,\dots, 0, & 0, & 0 \\ 0, & -q, & 1+q^2,\dots, 0, & 0, & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot \ \cdots \ \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0, & 0, & 0,\dots, 1+q^2, & -q, & 0 \\ 0, & 0, & 0,\dots, \ -q, & 1+q^2, & -q \\ 0, & 0, & 0,\dots, 0, & -q, & 1 \end{matrix} \right\|. \tag{12}$$

Hence, the elements $(S^{-1})_{ij}$ and $(K^{-1})_{ij}$ placed in i-th row and in j-th column of matrices S^{-1} and K^{-1}, respectively, can be written shortly as

$$(S^{-1})_{ij} = \begin{cases} 1 & \text{for } i = j = 0 \\ 1/\sqrt{1-q^2} & \text{for } i = j \neq 0 \\ -q/\sqrt{1-a^2} & \text{for } j = i+1, \text{ where } 0 \leqslant i \leqslant n-1 \\ 0 & \text{for } j > i+1 \text{ and } 0 \leqslant i < n-1 \\ 0 & \text{for } j < i, \text{ where } 1 \leqslant i \leqslant n \end{cases} \tag{13}$$

and

$$(K^{-1})_{ij} = \begin{cases} 1/[k_0^2(1-q^2)] & \text{for } i = j = 0, n \\ (1+q^2)/[k_0^2(1-q^2)] & \text{for } i = j \neq 0, n \\ -q/[k_0^2(1-q^2)] & \text{for } |i-j| = 1 \\ 0 & \text{for } |i-j| > 1, \end{cases} \tag{14}$$

where $i, j = 0, 1, 2, \ldots, n$.

The need of the inverse K^{-1} of the covariance matrix K arises when the calculations connected with the minimization of least squares are to be performed and when the simple correlation function (2) is the acceptable approximation to the actual correlation function of the disturbing random component of the treated data. As we see from Eq. (14), simple explicit expressions exist for the element of the inverse.

Reference

1. J. H. Laning and R. H. Battin, "Random Processes in Automatic Control," McGraw-Hill, New York, 1956.

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